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Engineering Notes

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Optimal Control of Target Tracking with State Constraints via Cell Mapping

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I. Introduction

THE cell mapping (CM) methods were first introduced by Hsu[1] to study the dynamics of nonlinear systems. CM methods have been used in various application areas such as fuzzy controls,[2] optimum trajectory planning in robotic systems,[3] optimal control of populations,[4] and semi-active control,[5] among others. Considerable work has been devoted to improving the CM accuracy and to relaxing its computational demands.[6,7] The method is most effective for low-dimensional systems, although there are some studies that extend it to higher dimensional problems.[8,9]

Many studies of the control problem of tracking and intercepting moving targets have been done, particularly in the areas of aerospace engineering and robotics. Extensive effort has been devoted to study pursuit–evasion problems in the so-called differential games.[10–13] CM methods have also been applied to the differential games.[14,15]

For strongly nonlinear dynamic systems, the optimal control problem of target tracking under state and control constraints is quite difficult to solve analytically or numerically. The present paper presents such a study for fixed and mobile targets by using the simple CM (SCM) method. The remainder of the Note is organized as follows. The search algorithm for fixed final state optimal control solutions is presented in Sec. II. Extensions of the SCM method to deal with mobile targets, state constraints, and target state estimations are presented in Sec. III. A discussion on the multilevel discretization of the control set is also presented. In Sec. IV, two examples are presented. First, the well-known minimum time-optimal control problem of a free particle in a one-dimensional space with state constraints and a mobile target is studied. Second, the minimum time navigation problem of a boat in a vortex field with a multilevel control set is studied for fixed and moving targets. Predictions of the future target path are included in the method of solution.

II. Optimal Control via CM

The description of the SCM method and the optimal control formulation in the context of SCM may be found in Refs. 4 and 7. In the SCM method, the trajectory of the system is approximated by the mappings between cell centers. This approximation can introduce large errors in long-term solutions. The error can lead to false optimal solutions with the SCM method.[3,16,17] An iterative nonuniform time-step CM method has been proposed by the authors in Ref. 7 to deal with this problem. The strategy relaxes the requirement for small cells, improves the method accuracy, and avoids degenerated solutions.

General Mapping Database

To apply the SCM method to the optimal control problem, we first construct a database of cell mappings under all controls in $\mathbb{U}$ by integrating the state equation. Let N_c denote the number of controls in $\mathbb{U}$. For every control and every cell center as initial condition, the equations of motion can be integrated until the first N_m consecutive cells are visited. Let $\boldsymbol{N}_m$ denote the set of mappings from a preimage cell to the first N_m consecutive image cells along the system trajectory under a given control. These mappings have nonuniform mapping time steps. The general control database contains the following elements: for each preimage cell z_i, there will be $N_c \times N_m$ image cells z_j, the corresponding controls $\boldsymbol{u}_l$, the associated mapping time steps Δt_{ijl}, and the incremental control costs

$$\Delta J_{ijl} = \int_{t_0}^{t_0 + \Delta t_{ijl}} L[\boldsymbol{x}(t), \boldsymbol{u}(t)]\, \mathrm{d}t$$

where z is a vector of integers denoting the location of a cell. We denote the complete set of mappings by $\boldsymbol{M}$. A special subset of $\boldsymbol{M}$ denoted by $\boldsymbol{N}$ contains the image cells of the closest neighbors of every preimage cell z_i under all controls in $\mathbb{U}$.

Algorithm

Let $\boldsymbol{\Omega}$ denote the set of cells representing the target set defined by $\boldsymbol{\Psi}[\boldsymbol{x}(T), T] = 0$. Let N_b be the number of backward search iterations that we would like to carry out. Initially, the search is over the mapping set $\boldsymbol{N}$ according to the following steps.

1) Search through all of the mappings to find all of the cells in the set that are mapped into $\boldsymbol{\Omega}$ in one step.

2) Assign a cumulative cost to the cells found in step 1. The cumulative cost is the smallest cost for the system to move from the current cell to the original target set. It is calculated by adding the cumulative cost of the image cell and the incremental cost of the current cell. If more than one image cell reaches the target set, the mapping with the smallest cumulative cost is taken.

3) Expand the target set $\boldsymbol{\Omega}$ by including the cells found in step 1 with the smallest cumulative cost only.

4) Repeat the search from step 1 until the initial condition is reached.

5) Examine the cumulative costs of all N_m consecutive image cells z_j for every preimage cell z_i and for every control $\boldsymbol{u}_l$ in $\mathbb{U}$. We retain the image cell that has the smallest cumulative cost. These cells are stored in a set $\boldsymbol{M}^*$.

6) Repeat from step 1 over the set $\boldsymbol{M}^*$ for $N_b - 1$ times.

The first four steps describe the usual CM approach for optimal control problems. The last two were developed to reduce possible false optimal solutions.[7] It has been shown that the average cost over the entire phase space is reduced with the number of backward searches over the set $\boldsymbol{M}^*$. The final set $\boldsymbol{M}^*$ contains the information on the location of the switching curves and the optimal controls for each cell. A discriminating function that considers smoothness and local continuity of the trajectories may be needed to break the cost ties in the backward search.

Some remarks on the algorithm are in order. This algorithm can be viewed as a backward implementation of the dynamic programming method over the cell state space. For nonlinear systems, the optimal control is in general a function of initial conditions. When we apply

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the algorithm to cover all of the cells that can be controlled to the target, we effectively obtain the global solution of the optimal control for all possible initial conditions.

III. Extensions

State Constraints

State constraints can be naturally included in the discrete cellular state space. For example, if a domain in the state space is not allowed to be accessed, sometimes known as the taboo zone, we can change all of the cells in the taboo zone to be sink cells for all admissible controls. A sink cell is mapped only to itself.[16] Hence, in the backward search, the sink cells will never be reached. Note that changing the dynamics of taboo cells does not affect the dynamics of the system in the rest of the region. This method of including the state constraints allows us to deal with several state constraints of different kinds including time-varying and multiply connected state domains.

Mobile Targets

The moving target problem can be solved by concatenating a sequence of fixed final state problems. For a given initial condition and target, the backward search algorithm finds an optimal control sequence that drives the system from its current state to the target. Because of the motion of the target, we may not execute the entire control sequence. After the target moves to a different location, a new backward search should be carried out to find another optimal control sequence.

When the target trajectory is completely unknown beforehand, we call this case the zero prediction horizon. The control at a given step is determined based on the current target location. In real-time implementation of the control with zero prediction horizon, there will be a time delay in the optimal control solution. When the target trajectory is known ahead of time, the backward search can be modified to account for the future location of the target to reduce the time delay.

Assume that we are able to predict the future locations of the target n clock cycles ahead from its present location. We call this case the prediction horizon n. Note that a clock cycle need not be the same as the mapping time step of the system. For convenience of discussion, we shall treat the clock cycle the same as the mapping time step. The backward search algorithm for optimal navigation can be stated as follows.

Set a counter $k = 1$ initially. Identify the cell as the target set $\boldsymbol{\Omega}$, where the target will be in k steps ($k \leq n$). A backward search starting from $\boldsymbol{\Omega}$ is carried out for k steps or until the current system state is reached, whichever takes less time. If the system state is not reached, we increase k by one. Find the cell where the target will be in the next step. Then, we perform a new backward search for k steps with this cell as the new target set $\boldsymbol{\Omega}$. This process will be repeated until $k = n$ if the current system state is not reached in the backward search. If the current system state is reached within $k \leq n$ steps, the entire sequence of the controls will be executed.

If $k = n$ and the current system state has not been reached yet, the backward search will continue until the system state is finally reached or until the target set of the algorithm is not expandable. This last condition occurs when the system is uncontrollable. Once the current system state is reached in the backward search, the optimal control sequence can be executed for one or a few steps. For better tracking performance, the optimal control sequence should be executed for n or less steps.

After the solution is executed and the target moves on, the new system state and target location define the initial and final conditions of a new problem that can be solved in the same manner. Note that if the optimal control is executed for only one step before it is updated again by the search algorithm, it will provide the best performance at the expense of more searches. If the backward search algorithm is implemented in real time, the control performance and the searching time need to be compromised.

Multilevel Controls

When multiple levels of controls with small differences or short mapping times are used, different controls may map one cell to a same image cell. Bigger time steps or smaller cells can be used to avoid this problem. However, bigger time steps will imply that the control will switch over longer time intervals, leading to a narrower bandwidth. It turns out that the mapping time step, the cell size, and the level of discretization of the control set are all closely related. A proper combination of these three parameters can lead to efficient and accurate solutions. Further discussion of this aspect will be presented elsewhere.

IV. Numerical Examples

One-Dimensional Motion Control of a Particle

Consider the well-known bang–bang control problem of moving a unit point mass from any initial condition to a final state in minimum time. The state equation of the system is given by

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \boldsymbol{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{1}$$

where u is the control. A problem that considers a mobile target in a constrained state spaced is presented: 2025 square cells are used to cover the region $\Theta = [-1.5, 1.5] \times [-1.5, 1.5]$ centered around the origin of the phase plane. We have chosen $N_m = 2$, and the admissible control set $\mathbb{U} = \{1, -1\}$, that is, the scalar control input u is a unit force that can take either a value of 1 or -1. The state vector $\boldsymbol{x} = (x_1, x_2)^T$ consists of the displacement and velocity. We denote a taboo zone Λ as the interior state region $[-0.1, 0.1] \times [0.6, 1.75]$. Here, Λ is a purely mathematical entity. The initial condition for the system is taken as $\boldsymbol{x}_0 = (-1.45, 0)^T$. The control objective is to drive the system from $\boldsymbol{x}_0$ to the target in minimum time without crossing the taboo zone Λ. Figure 1 shows the results of this problem.

Optimal Navigation in a Vortex

Consider a boat moving on the (x_1, x_2) plane with constant velocity relative to the water. Let the control u be the heading angle with respect to the horizontal shore. A set of 12 evenly spaced angles values over $[-\pi, \pi]$ is taken as the control set $\mathbb{U}$. The velocity field of a vortex is centered at (0, 0). The tangential velocity v of the water is given by $v = ar(b - r)/e^{cr}$ where r is the distance to the center of the vortex, and a, b, and c are constants. The motion of the boat is governed by:

$$\dot{x}_1 = \cos(u) - v(x_2/r), \qquad \dot{x}_2 = \sin(u) + v(x_1/r) \tag{2}$$

where (x_1, x_2) represents the location of the boat. The region $\Theta = [-1, 1] \times [-1, 1]$ is discretized with 7921 square cells. We have also chosen $N_m = 1$. First, we consider a fixed final state problem.

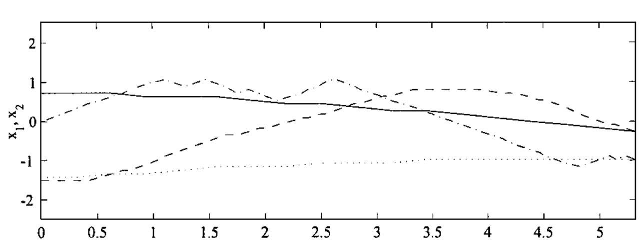

Fig. 1a Position and velocity of the particle (– – –, –·–) and the target (——, · · · · ·).

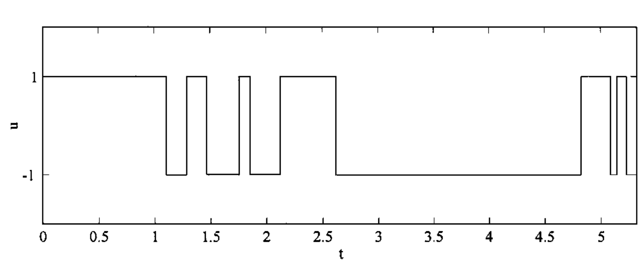

Fig. 1b Optimal control.

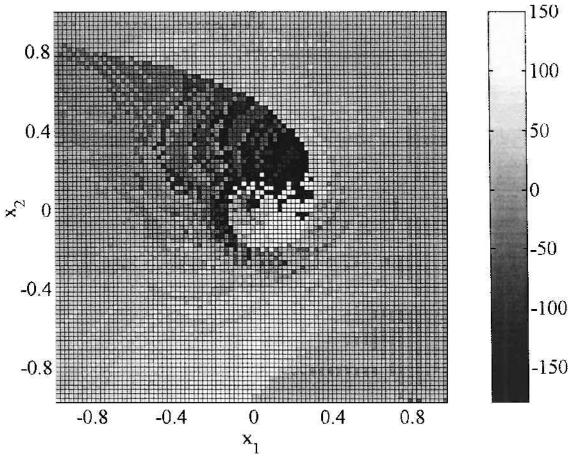

Fig. 2 Distribution of optimal angles as a function of initial conditions for navigating the boat to the left side of the vortex.

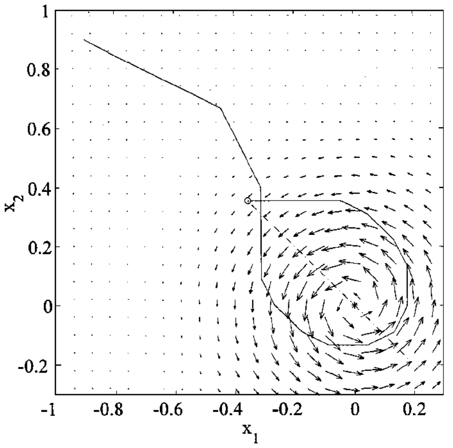

Fig. 3 Trajectory of the boat (——) and the target (-·-), terminal condition where the boat reaches the target (○).

The objective is to drive the boat around the vortex in minimum time starting from an initial location $\boldsymbol{x}_0$. The target is defined by the set $\boldsymbol{\Psi} = \{(x_1 = 1, x_2)\}$. Let $\boldsymbol{\Omega}$ be the set of cells that form a discrete representation of $\boldsymbol{\Psi}$. Specifically, $\boldsymbol{\Omega}$ consists of the cells in the right column of Θ. The global distribution of the optimal navigation angles as a function of the initial location (x_1, x_2) is shown in Fig. 2. Every cell is colored according to the starting optimal angle of the optimal control sequence. As expected, the solution indicates that under the given conditions the entire phase space is controllable. That is, we can navigate the boat to the target from any initial condition $\boldsymbol{x}_0 \in \Theta$. Figure 2 shows a discontinuity of the distribution of optimal controls. On the right side of the discontinuity the optimal control drives the system against the current of the vortex to reach the target. On the left side, the optimal control directs the boat along the streamlines of the vortex. Note that the distribution of the cumulative cost as a function of the initial condition is continuous despite the discontinuity of the control. The dependence of optimal controls on the initial state for such nonlinear systems has not been extensively studied in the literature. As we have demonstrated here, the CM method offers an attractive tool for studying this problem.

Next, we consider the problem of tracking a moving target. The target is assumed to cross the vortex through its center along a straight path while the boat is at $\boldsymbol{x}_0 = (-0.9, 0.9)$ initially. It has been found that for a control strategy based on the current location of the target only leads the boat to circle twice around the center of the vortex in 1.7 time units before hitting the target. Figure 3 shows the trajectory of the boat under control with prediction horizon two. The vector field of the vortex is also shown. The time to reach the target is now 0.9 time units. When the prediction horizon is infinite, the time to reach the target is 0.7 time units.

V. Conclusions

We have presented a study of optimal control problems of reaching fixed or moving targets via the SCM method. Extensions to the CM method are presented for problems involving state constraints and mobile targets. A computational approach is proposed to incorporate predictions of the target trajectory in the optimal solution. Two examples have been studied to demonstrate the method. We found the existence of a discontinuity of the optimal control as a function of initial conditions in the navigation problem. Such a global property of the optimal control can be readily obtained by the CM method and is otherwise very difficult to obtain. This example has also indicated that the prediction of the target trajectory ahead of time can reduce the target chasing time. The current method has a potential for real-time application. The method is most effective for low-dimensional systems, although there are some studies that extend it to higher dimensional problems.

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